Data structures for 3D Meshes

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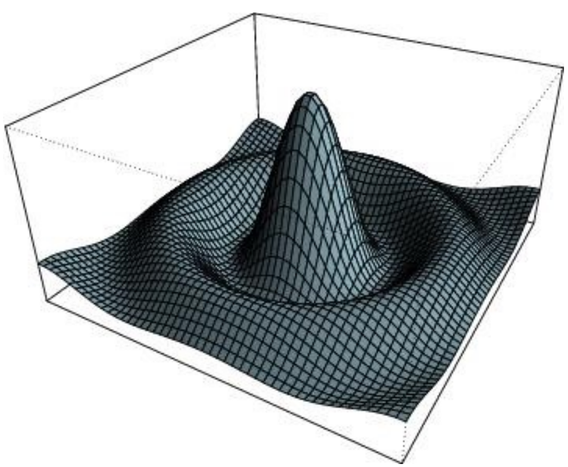
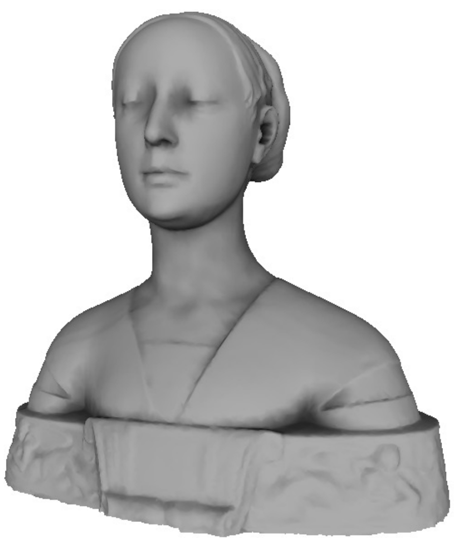
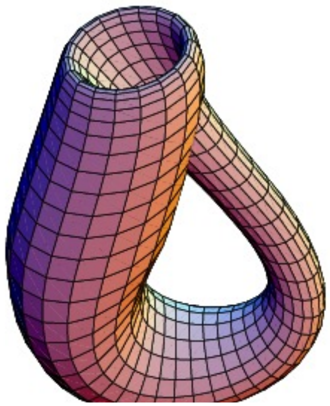
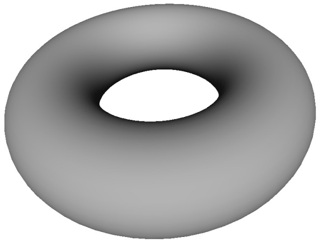
Surfaces

- A 2-dimensional region of 3D space
- *A portion of space having length and breadth but no thickness*









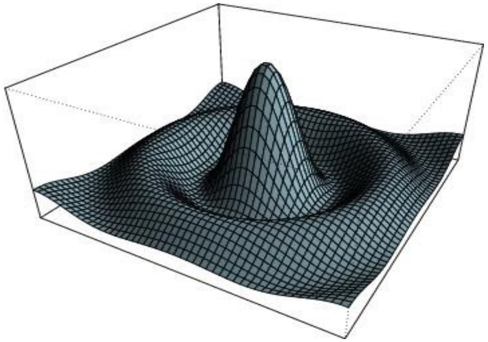
Defining Surfaces

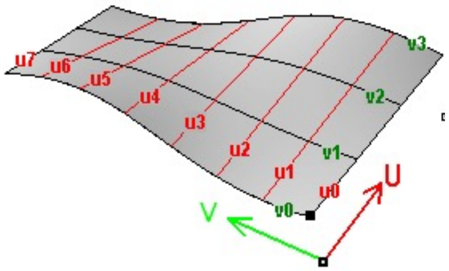
- Analytic definitions
  (aka exact)
  - **Parametric surfaces**
    A function that maps points on a 2D domain over a 3D surface

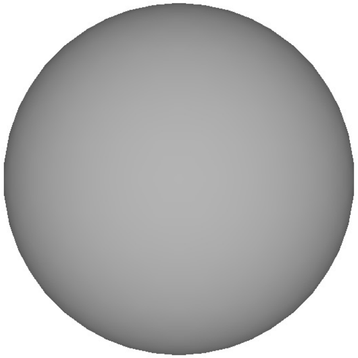
$$S: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

  - **Implicit surfaces**
    A surface defined where the points of the 3D space satisfy a certain property
    (usually a given function = 0)

$$S = \{p \in \mathbb{R}^3 : f(p) = 0\}$$





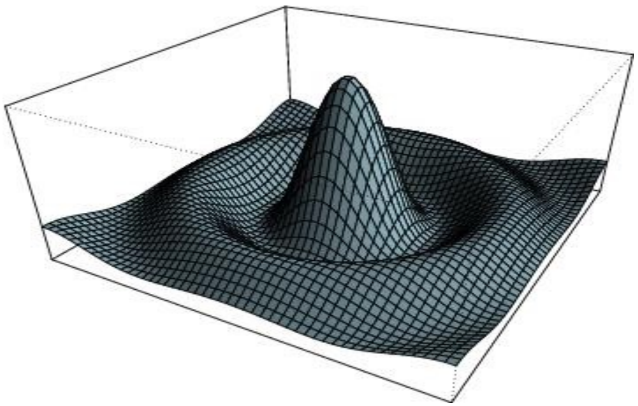


Analytic Surfaces

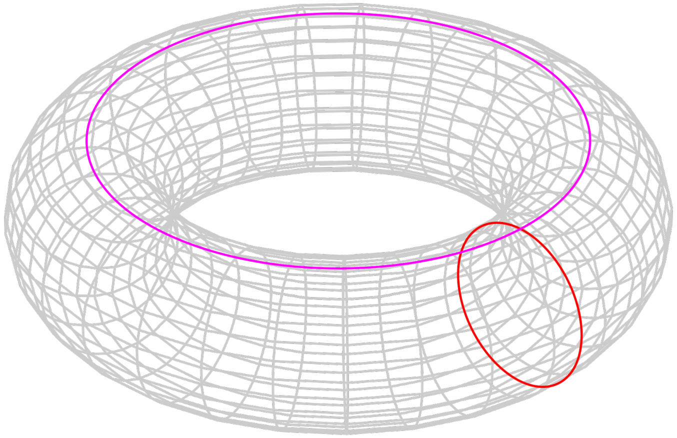
❖ Parametric surfaces

A function that maps points on a 2D domain over a 3D surface:

$$S: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$



$$S(x, y) = \left(x, y, \sin\left(\sqrt{(x^2 + y^2)}\right)/\sqrt{(x^2 + y^2)}\right)$$



$$x = (R + r \cdot \sin t) \cdot \cos s$$
$$y = (R + r \sin t) \cdot \sin s$$
$$z = r \cdot \cos t$$

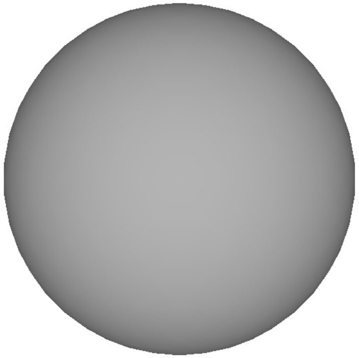
Analytic Surfaces

❖ Implicit surfaces

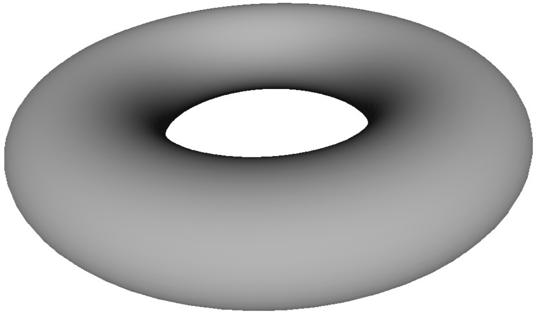
A surface defined where the points of the 3D space satisfy a certain property (usually a given function = 0)

$$S = \{p \in \mathbb{R}^3 : f(p) = 0\}$$

$S = \{(x, y, z): x^2 + y^2 + z^2 - r^2 = 0\}$



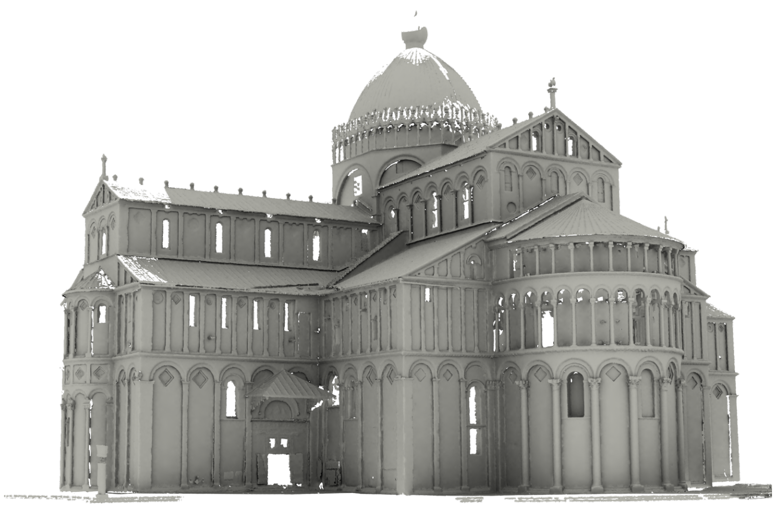
$S = \{(x, y, z): (x^2 + y^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2) = 0\}$



Representing Real World Surfaces

- Analytic definition falls short of representing *real world* surfaces in a *tractable* way

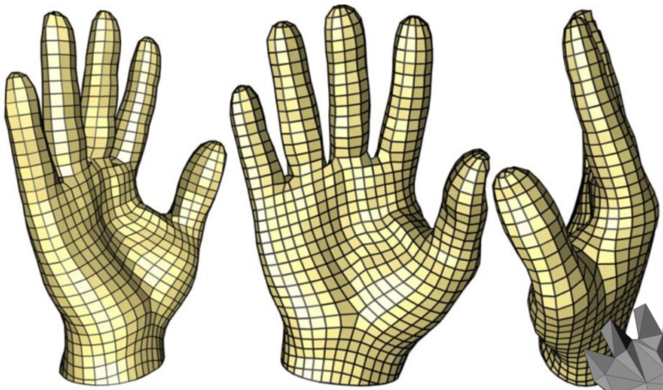
$$S(x, y) = ...?$$



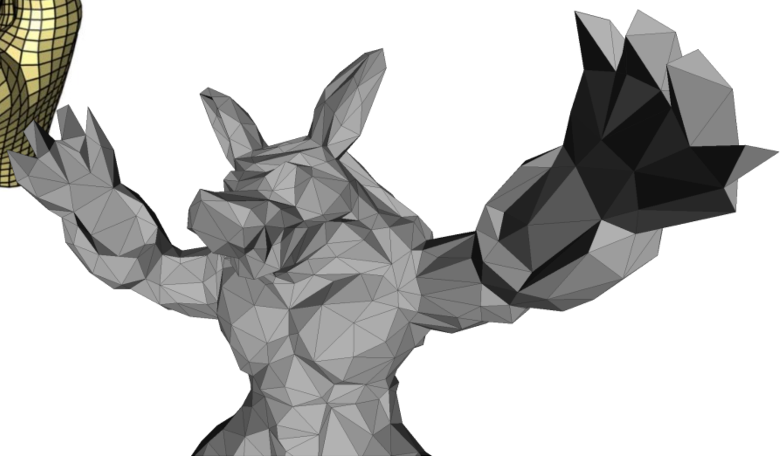
... surfaces can be represented by ***cell complexes***

Cell complexes (meshes)

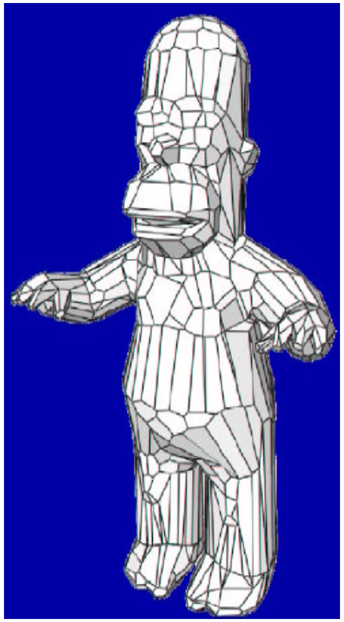
- Intuitive description: a continuous surface divided in polygons



quadrilaterals (quads)



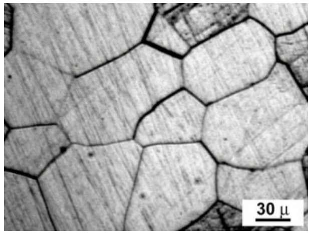
triangles

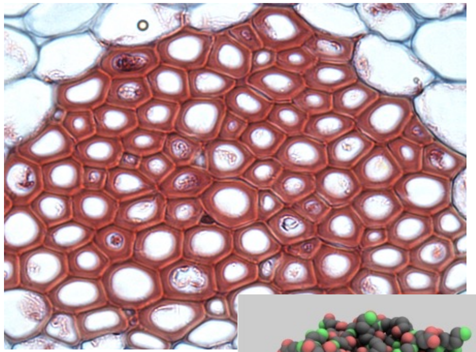
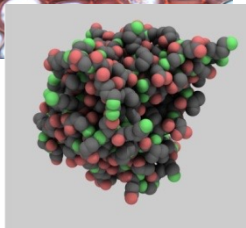


Generic polygons

Cell Complexes (meshes)

- In nature, meshes arise in a variety of contexts:
  - Cells in organic tissues
  - Crystals
  - Molecules
  - …
  - Mostly *convex* but *irregular* cells
  - Common concept: *complex* shapes can be described as *collections* of *simple building blocks*



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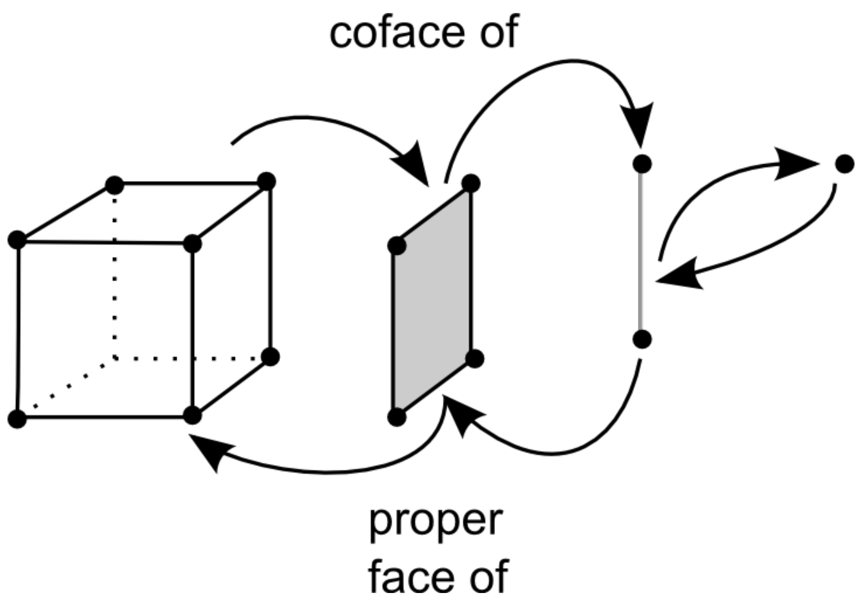
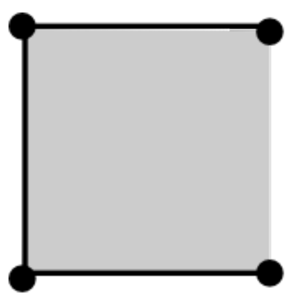
Cell Complexes (meshes)

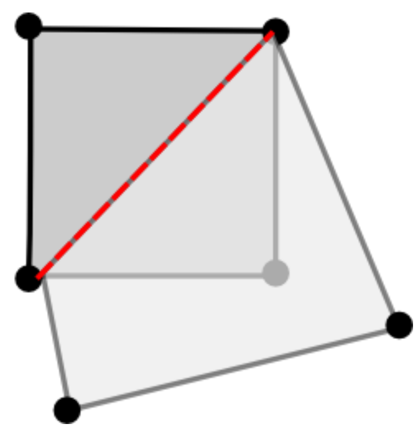
- Slightly more formal definition
  - a *cell* is a convex polytope in
  - a *proper face* of a cell is a lower dimension convex polytope subset of a cell



Cell Complexes (meshes)

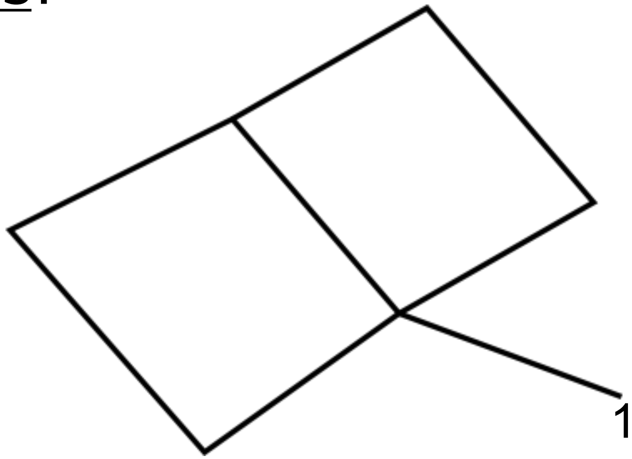
- a collection of cells is a complex **iff**
  - every face of a cell belongs to the complex
  - For every cells C and C′, their intersection either is empty or is a common face of both





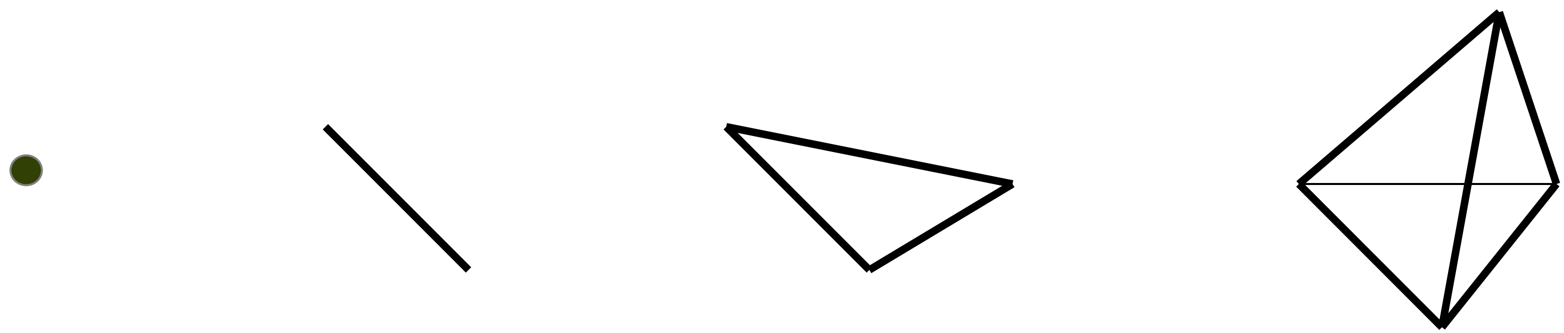
Maximal Cell Complex

- the **order** of a cell is the number of its sides (or vertices)
- a complex is a **k-complex** if the maximum of the order of its cells is *k*
- a cell is **maximal** if it is not a face of another cell
- a k-complex is **maximal** *iff* all maximal cells have order k
- short form : no dangling edges!



Simplicial Complex

- A cell complex is a **simplicial complex** when the cells are simplexes
- A ***d-simplex*** is the convex hull of *d+1* points in



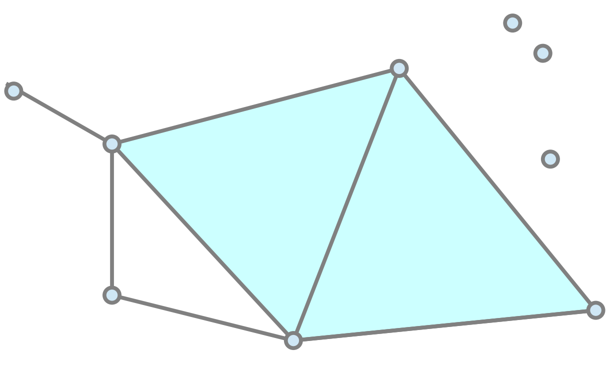
0-simplex 1-simplex 2-simplex 3-simplex

Sub-simplex / face

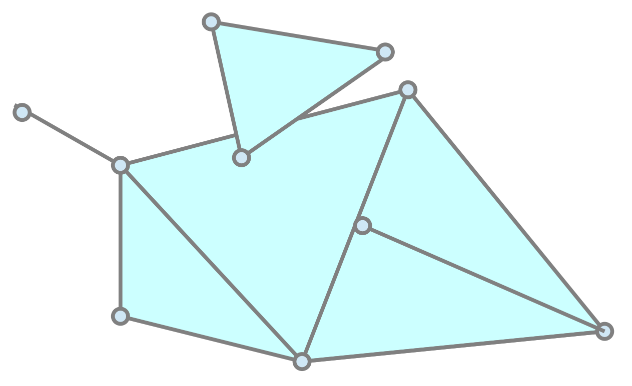
- A simplex σ' is called *face* of another simplex σ if it is defined by a subset of the vertices of σ
-
- If $\sigma \neq \sigma'$ it is a proper face

Simplicial Complex

- A collection of simplexes Σ is a simplicial k-complex iff:
  - $\forall\ \sigma_{1,}\ \sigma_{2,} \in \Sigma$
    $\sigma_1 \cap \sigma_2 \neq \emptyset \quad \Rightarrow \quad \sigma_1 \cap \sigma_2$ is a simplex of Σ
  - $\forall\ \sigma\ \in \Sigma$ all the faces of σ belong to Σ
  - k is the maximum degree of simplexes in Σ



OK

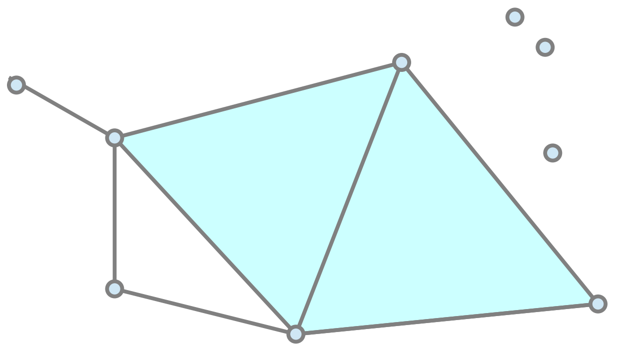


Not Ok

Simplicial Complex

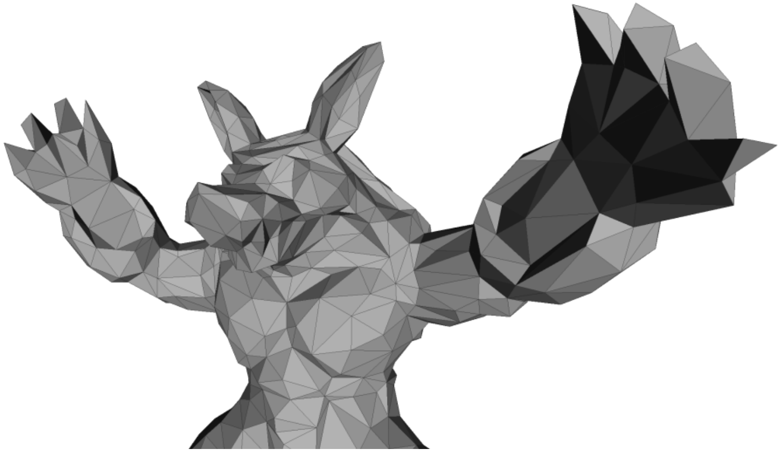
- A simplex σ is maximal in a simplicial complex Σ if it is not a proper face of a another simplex σ' of di Σ
- A simplicial k-complex Σ is maximal if all its maximal simplex are of order k
  - No dangling lower dimensional pieces

Non maximal 2-simplicial complex



Meshes, at last

- When talking of *triangle mesh* the intended meaning is a **maximal 2-simplicial complex**



Topology vs Geometry

- It is quite useful to discriminate between:
  - Geometric realization
    - **Where** the vertices are actually placed in space
  - Topological Characterization
    - **How** the elements are combinatorially connected

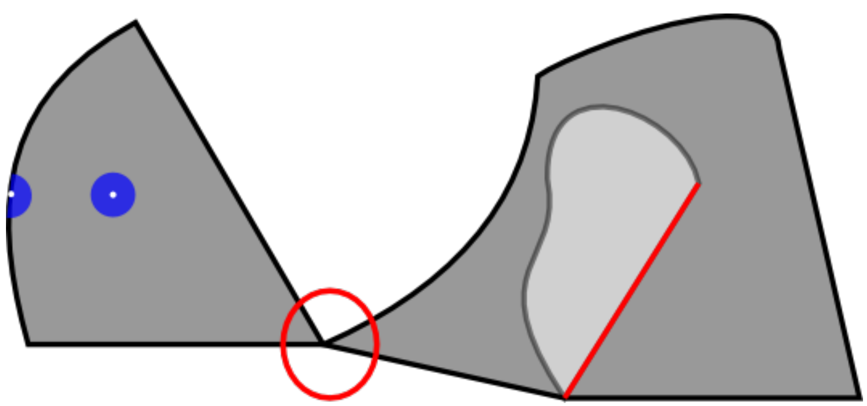
Topology vs geometry 2

Given a certain shape we can represent it in many different ways; topologically different but quite similar from a geometric point of view (demo klein bottle)

- Note that we can say many things on a given shape just by looking at its topology:
  - Manifoldness
  - Borders
  - Connected components
  - Orientability

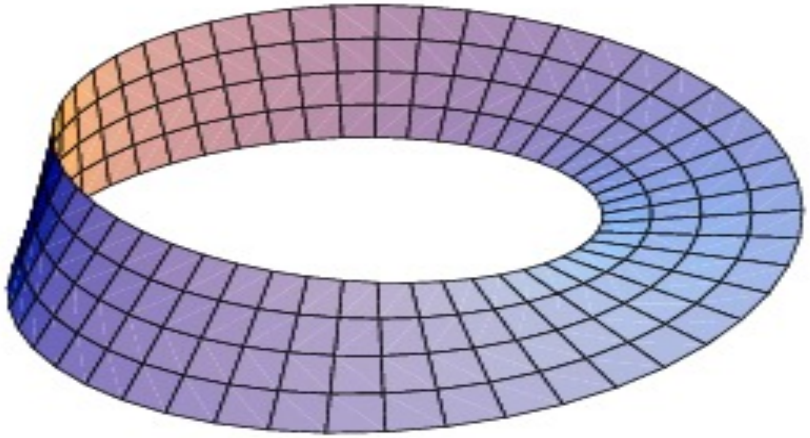
Manifoldness

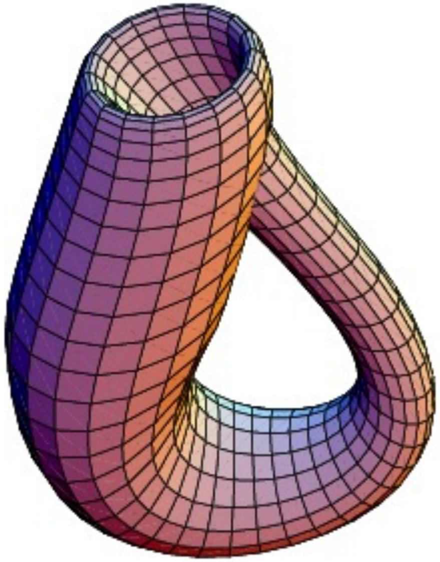
- a surface S is **2-manifold** *iff:*
  - the neighborhood of each point is homeomorphic to Euclidean space in two dimension
    or … in other words..
  - the neighborhood of each point is homeomorphic to a disk (or a semidisk if the surface has boundary)



Orientability

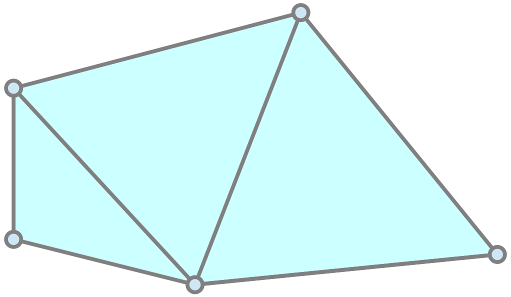
- A surface is **orientable** if it is possible to make a consistent choice for the normal vector
  - …it has two sides
- Moebius strips, klein bottles, and non manifold surfaces are not orientable





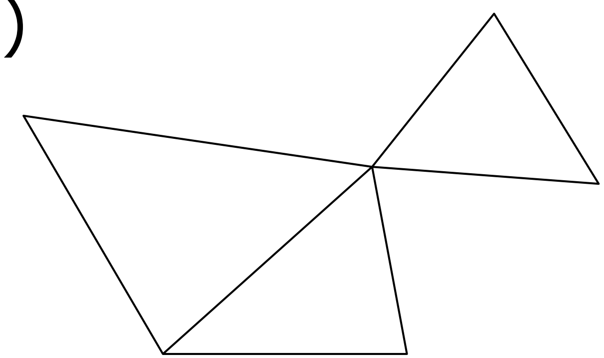
Adjacency/Incidency

- Two simplexes σ e σ' are **incident** if σ is a proper face of σ' (or viceversa)
- Two k-simplexes σ e σ' s are **m-adjacent** (k>m) if there exists a m-simplex that is a proper face of σ e σ'
  - Two triangles sharing an edge are 1-adjacent
  - Two triangles sharing a vertex are 0-adjacent



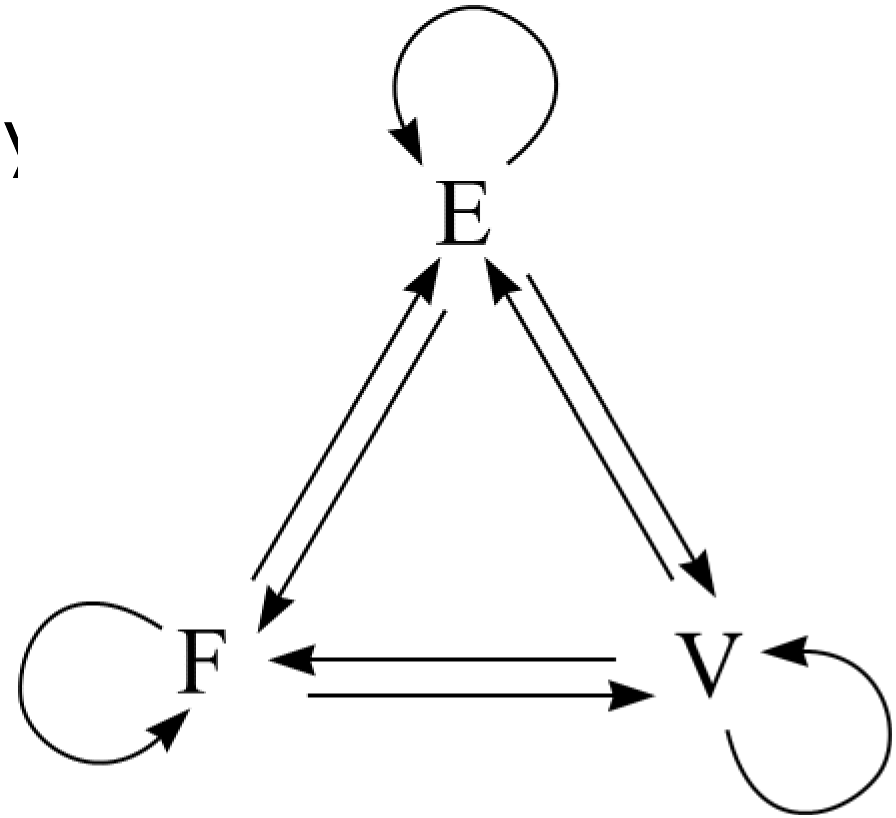
Adjacency Relations

- An intuitive convention to name practically useful topological relations is to use an *ordered* pair of letters denoting the involved entities:
  - **FF** edge adjacency between triangular **F**aces
  - **FV** from Faces to Vertices (e.g. the vertices composing a face)
  - **VF** from a vertex to a triangle (e.g. the triangles incident on a vertex)



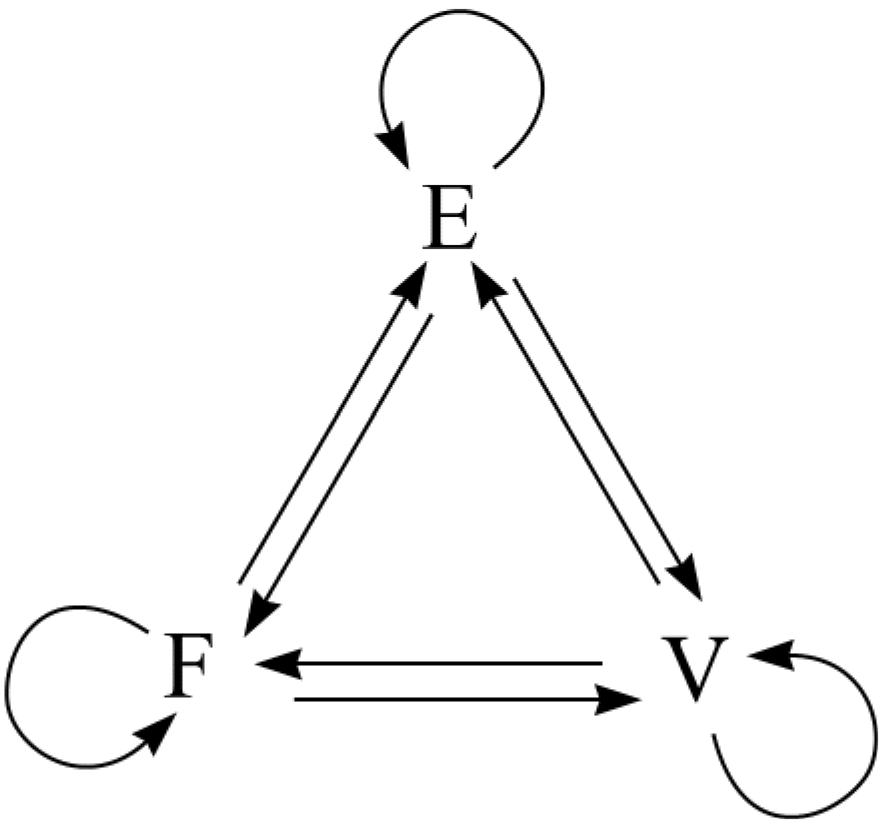
Adjacency Relationship

- Usually we only keep a small subset of all the possible adjacency relationships
- The other ones are procedurally generated



Adjacency Relation

- FF ~ 1-adjacency
- EE ~ 0 adjacency
- FE ~ proper subface of F with dim 1
- FV ~ proper subface of F con dim 0
- EV ~ proper subface of E con dim 0
- VF ~ F in Σ : V proper subface of F
- VE ~ E in Σ : V proper subface of E
- EF ~ F in Σ : E proper subface of F
- VV ~ V' in Σ : it exists an edge E:(V,V')



Partial adiacency

- For sake of conciseness it can be useful to keep only a partial information
  - VF* memorize only a reference from a vertex to a face and then surf over the surface using FF to find the other faces incident on V

Adjacency Relation

- For a two manifoldsimplicial 2-complex in R3
  - FV FE FF EF EV have bounded degree (are constant if there are no borders)
    - |FV|= 3 |EV| = 2 |FE| = 3
    - |FF| <= 2
    - |EF| <= 2
  - VV VE VF EE have variable degree but we have some avg. estimations:
    - |VV|~|VE|~|VF|~6
    - |EE|~10
    - F ~ 2V

Euler characteristic

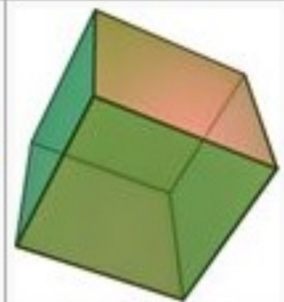
- $$\chi = V - E + F$$

V : number of vertices

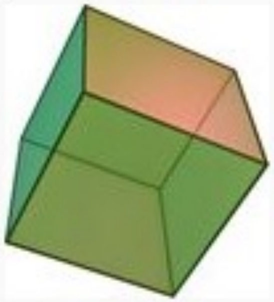
E : number of edges

F : number of faces

The Five Platonic Solids

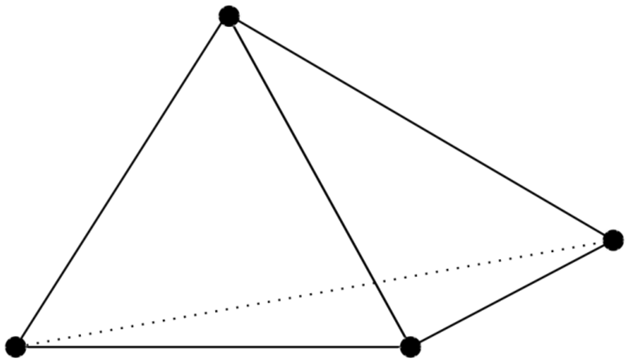
Tetrahedron	
Hexahedron or cube	
Octahedron	
Dodecahedron	
Icosahedron	

The Five Platonic Solids

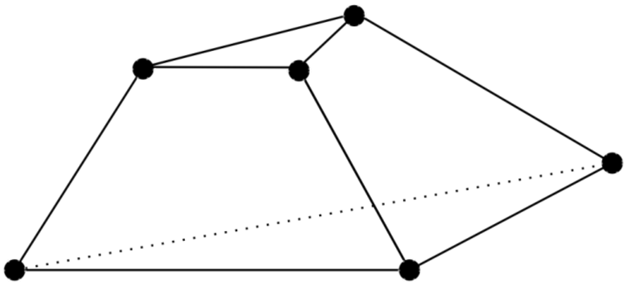
Tetrahedron		4	6	4
Hexahedron or cube		8	12	6
Octahedron		6	12	8
Dodecahedron		20	30	12
Icosahedron		12	30	20

Euler characteristics

- χ = 2 for any *simply connected* polyhedron
- proof by construction…
- play with examples:



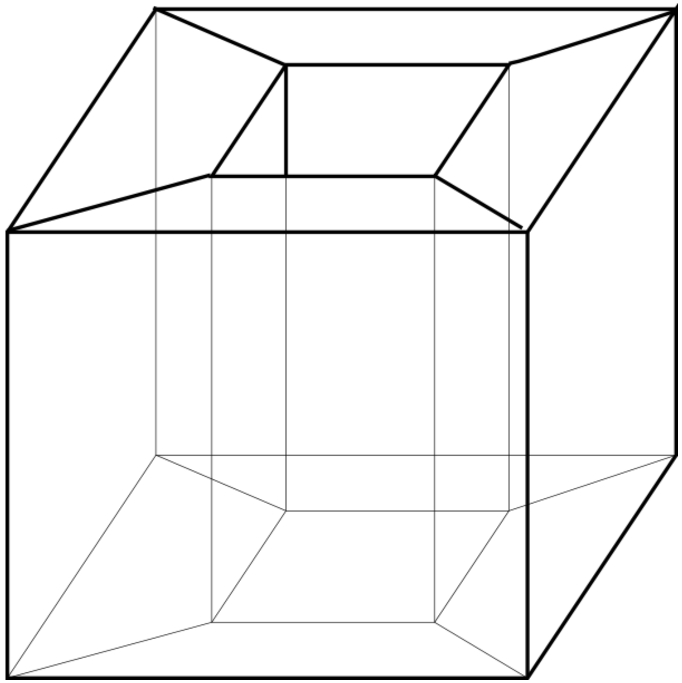
$$\chi = V - E + F$$
$$\chi = 4 - 6 + 4 = 2$$



$$\chi = (V + 2) - (E + 3) + (F + 1) =$$
$$\chi = (4 + 2) - (6 + 3) + (4 + 1) = 2$$

Euler characteristics

- let's try a more complex figure…

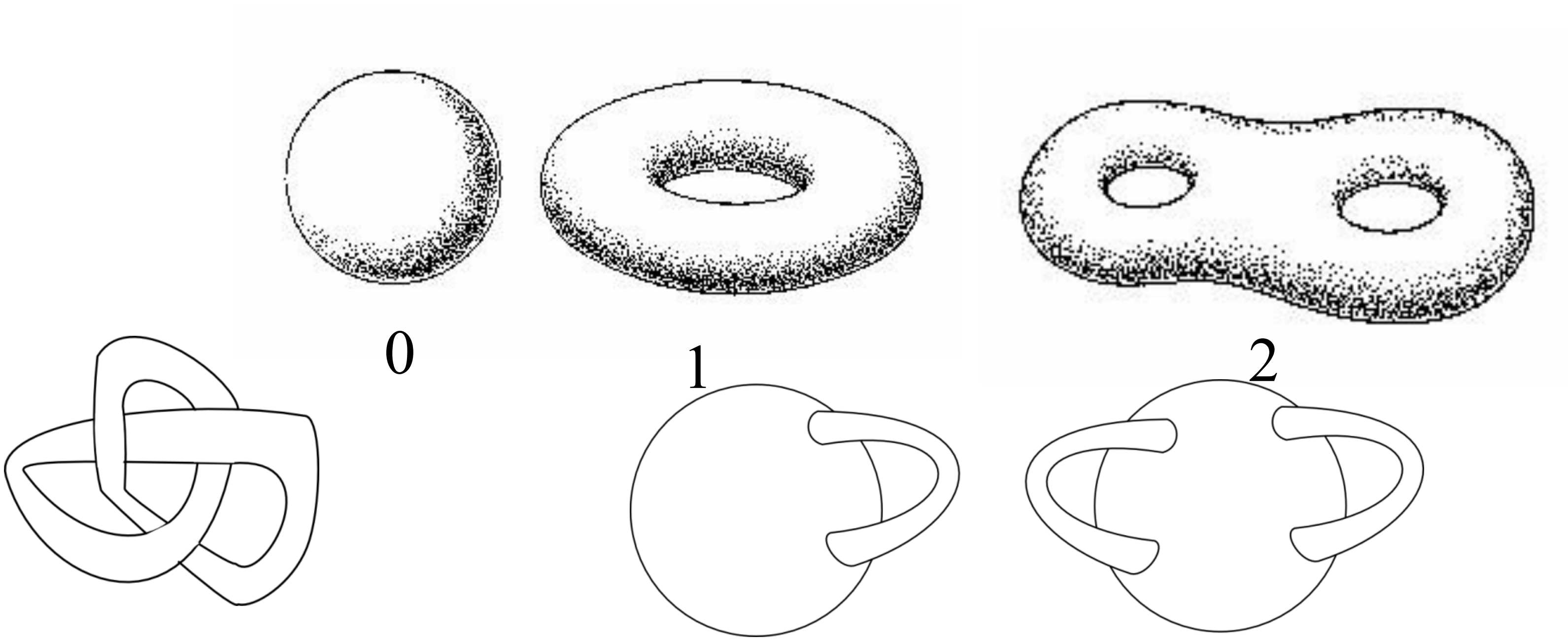


$$\chi = V - E + F$$
$$\chi = 16 - 32 + 16 = \mathbf{0}$$

- why =0 ?

Genus

- The **Genus** of a closed surface, orientable and 2-manifold is the maximum number of cuts we can make along non intersecting closed curves without splitting the surface in two.



- …also known as the number of *handles*

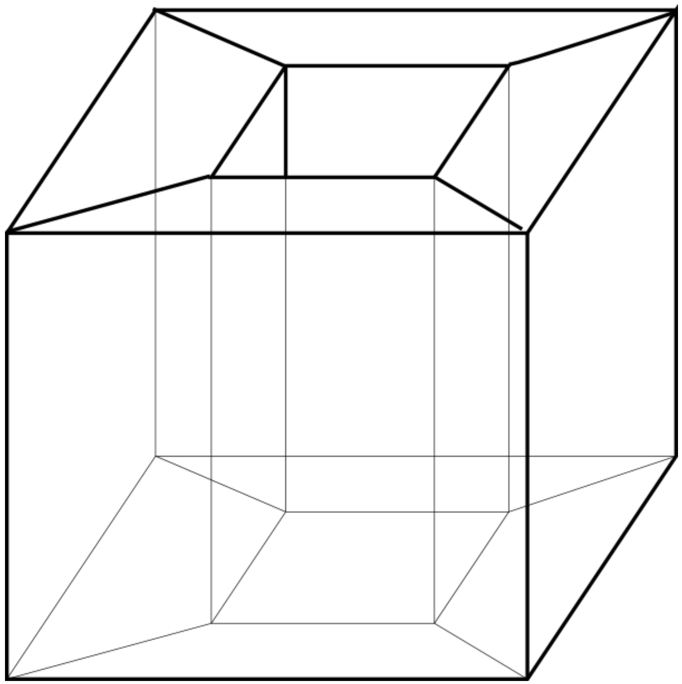
Genus

*To a topologist, a coffee **cup** and a **donut** are the same thing*

Euler characteristics

$$\chi = 2 - 2g$$

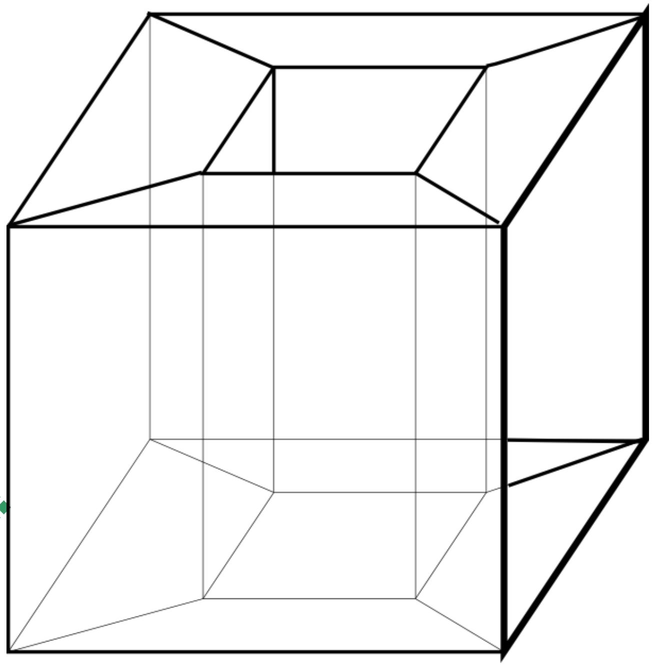
- where g is the genus of the surface



$$\chi = V - E + F$$
$$\chi = 16 - 32 + 16 = 0 = 2 - 2g$$

Euler characteristics

- let's try a more complex figure…remove a face. The surface is not closed anymore



$$\chi = V - E + F$$
$$\chi = 16 - 32 + 15 = -\mathbf{1}$$

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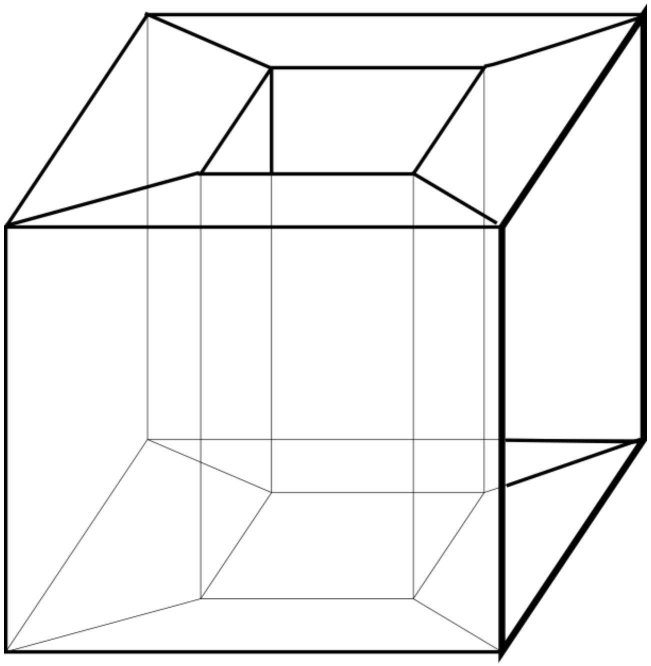
- why =-1 ?

Euler characteristics

$$\chi = 2 - 2g - b$$

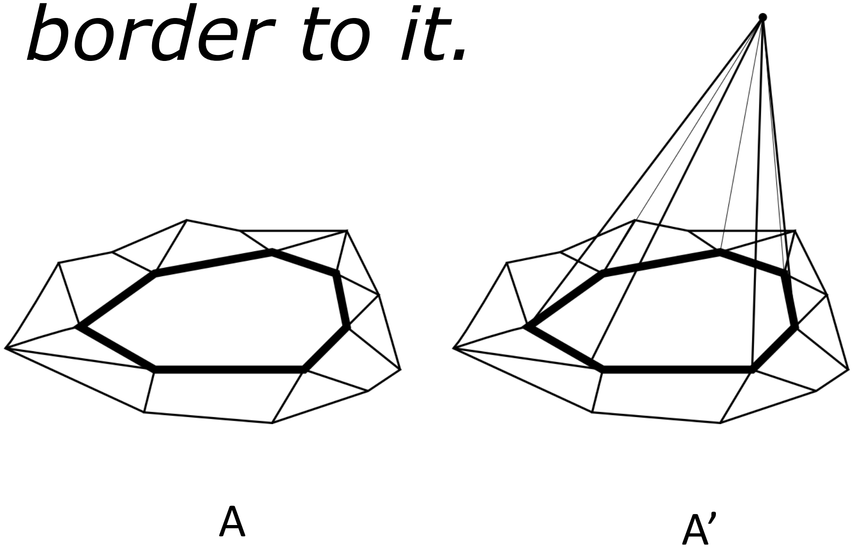
- where *b* is the number of borders of the surface



$$\chi = V - E + F$$
$$\chi = 16 - 32 + 15 = -1 = \boldsymbol{2 - 2g - b}$$

Euler characteristics

- *Remove the border by adding a new vertex and connecting all the* ***k*** *vertices on the border to it.*



$$X' = X + V' - E' + F' = X + 1 - k + k = X + 1$$

Converting Representations

Parametric Surface to Mesh

- *Easy. Just Sample the function on a regular domain and build a grid*

- *Issues*
- *Regular sampling does not imply regular meshing*

Converting Representations

Implicit Representation to Mesh

$$S = \{p \in \mathbb{R}^3 : f(p) = 0\}$$ $$S = \{p \in \mathbb{R}^3 : f(p) = 0\}$$

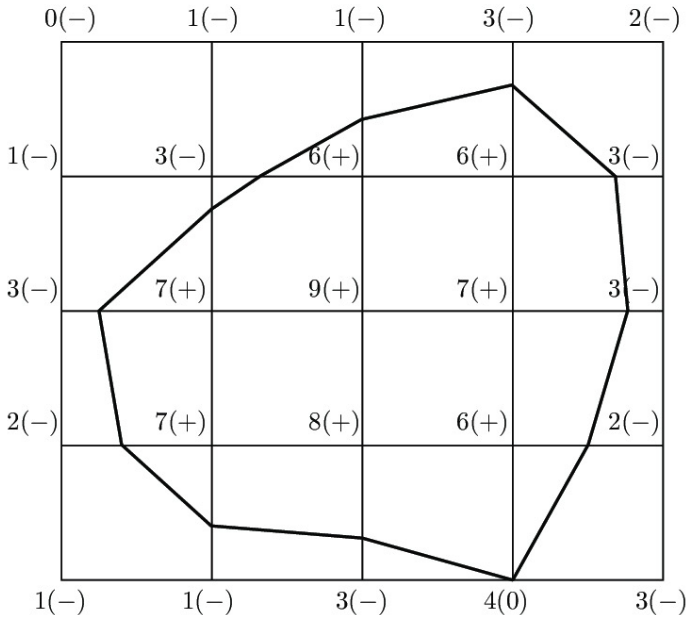
Isosurface on a regular grid

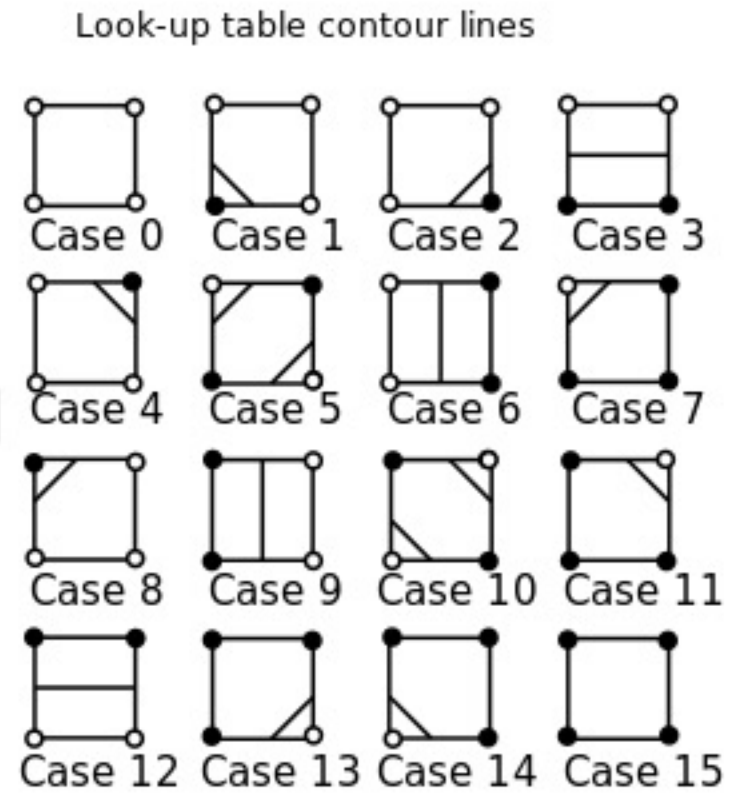
- *Sample the function on a regular grid and apply marching cube algorithm*

Converting Representations

Implicit Representation to Mesh

Marching Cube

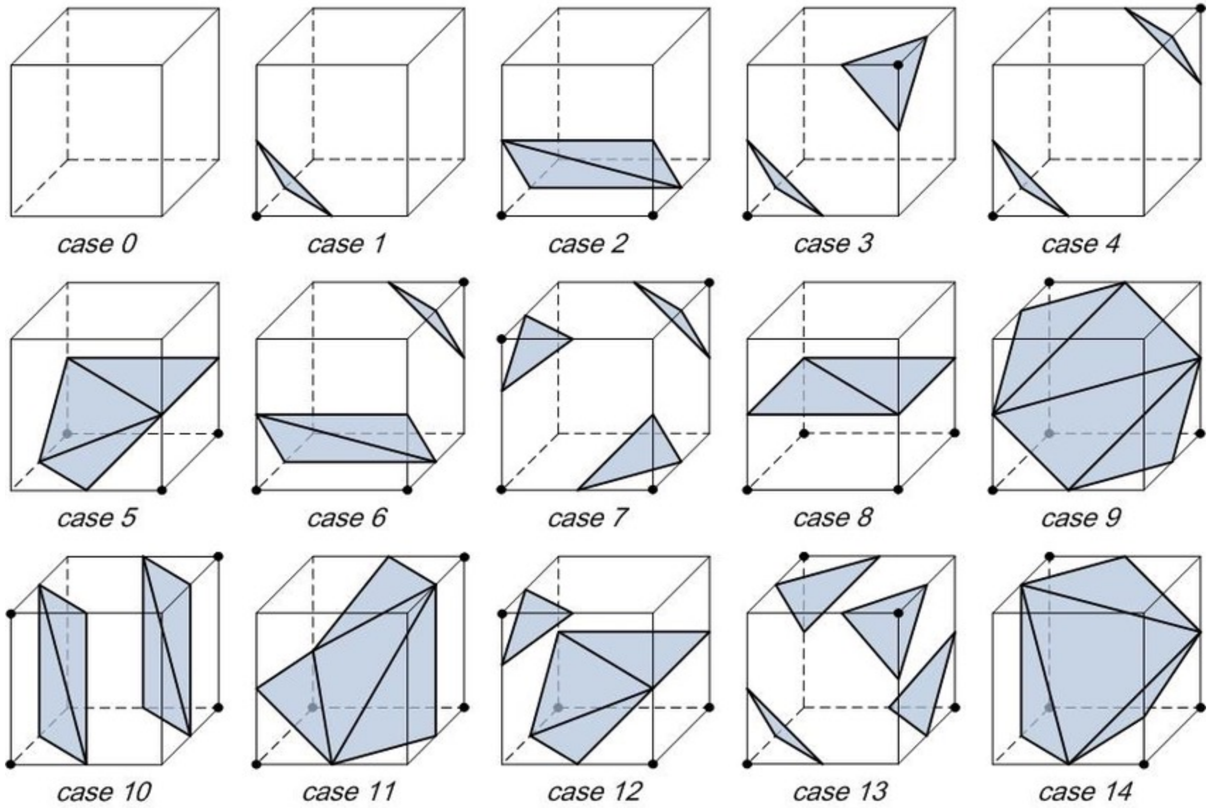




Converting Representations

Implicit Representation to Mesh

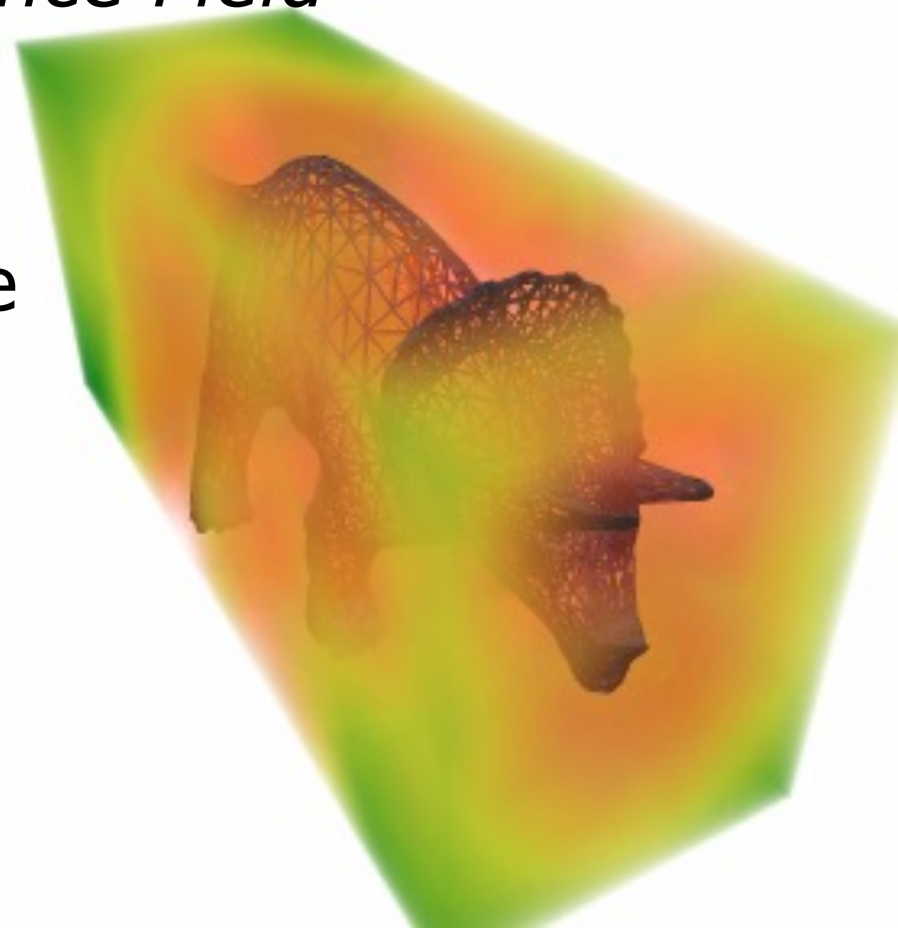
Marching Cube



Converting Representations

Mesh to Implicit Representation

Regularly Sampled Distance Field



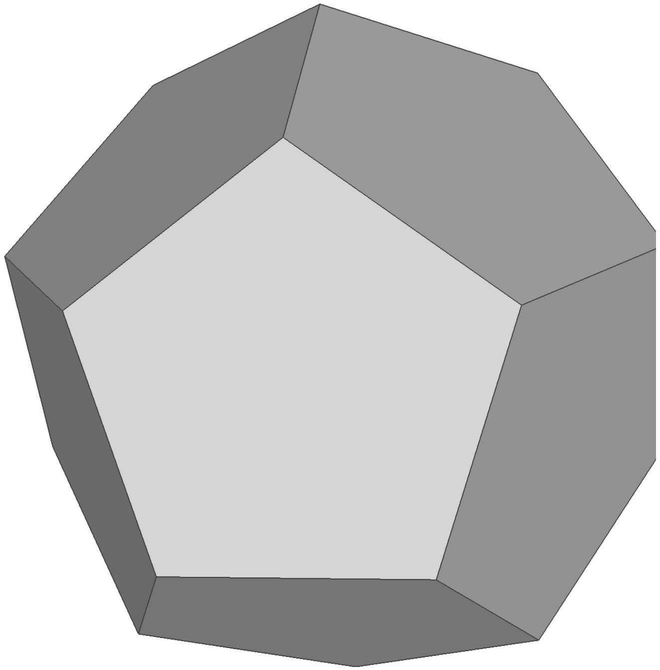
For each point on a grid store the signed distance from the surface

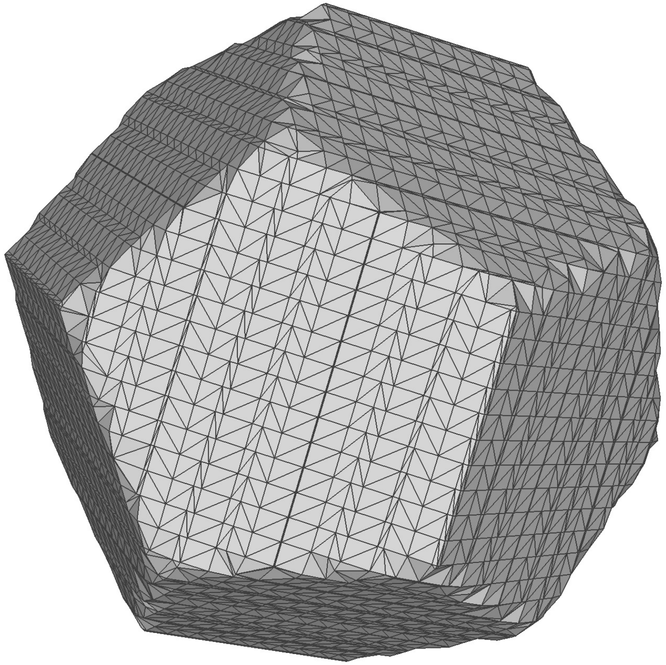
Converting Representations

Implicit Representation <-> Mesh

Issues:

- *Sampling Artifacts*





Mesh Data structures

- How to store geometry & connectivity?
  - compact storage
    - file formats
  - efficient algorithms on meshes
    - identify time-critical operations
    - all vertices/edges of a face
    - all incident vertices/edges/faces of a vertex

Face Set (STL)

- face:
  - 3 positions

Triangles		
x_{11} y_{11} z_{11}	x_{12} y_{12} z_{12}	x_{13} y_{13} z_{13}
x_{21} y_{21} z_{21}	x_{22} y_{22} z_{22}	x_{23} y_{23} z_{23}
...	...	...
x_{F1} y_{F1} z_{F1}	x_{F2} y_{F2} z_{F2}	x_{F3} y_{F3} z_{F3}

36 B/f = 72 B/v

no connectivity!

Typical Mesh Operation

- Access to individual vertices, edges, and faces. (enumeration of all elements in unspecified order)
- Oriented traversal of the edges of a face, which refers to finding the next edge (or previous edge) in a face.
- Access to the incident faces of an edge. Depending on the orientation, this is either the left or right face in the manifold case.
- Given an edge, access to its two endpoint vertices.
- Given a vertex, at least one incident face or edge must be accessible. Then for manifold meshes all other elements in the so-called one-ring neighborhood of a vertex can be enumerated (i.e., all incident faces or edges and neighboring vertices).

Shared Vertex (OBJ, OFF)

- vertex:
  - position
- face:
  - vertex indices

Vertices
x_1 y_1 z_1
. . .
x_V y_V z_V

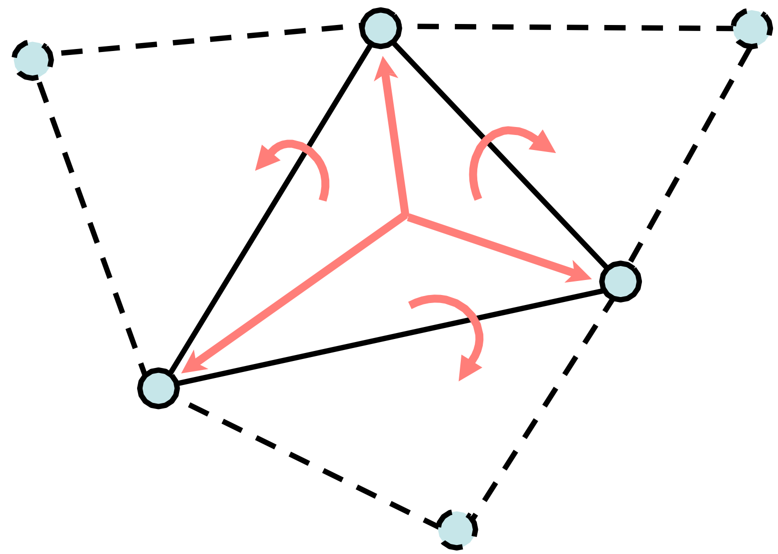
Triangles
v_{11} v_{12} v_{13}
. . .
. . .
. . .
. . .
v_{F1} v_{F2} v_{F3}

12 B/v + 12 B/f = 36 B/v

no neighborhood info

Face-Based Connectivity

- vertex:
  - position
  - 1 face
- face:
  - 3 vertices
  - 3 face neighbors

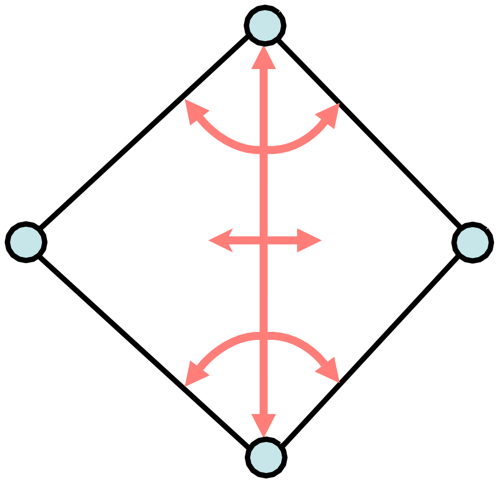


64 B/v

no edges!

Edge-Based Connectivity

- vertex
  - position
  - 1 edge
- edge
  - 2 vertices
  - 2 faces
  - 4 edges
- face
  - 1 edge

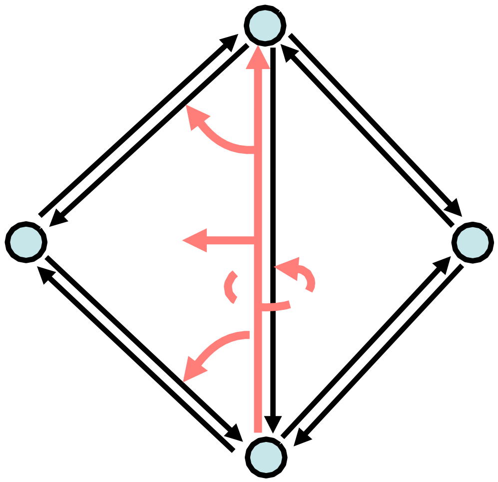


120 B/v

edge orientation?

Halfedge-Based Connectivity

- vertex
  - position
  - 1 halfedge
- halfedge
  - 1 vertex
  - 1 face
  - 1, 2, or 3 halfedges
- face
  - 1 halfedge



96 to 144 B/v

no case distinctions during traversal